MANIFOLDS WHICH ARE JOINS

BY

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1. **Introduction.** The *join* $A \circ B$ of topological spaces A and B is defined to be the disjoint union $A \cup (A \times B \times I) \cup B$ with each $(a, b, 0)$ of $A \times B \times I$ identified to $a \in A$ and each $(a, b, 1)$ of $A \times B \times I$ to $b \in B$(2). It is a consequence of the product and identification topologies that A and B are imbedded in $A \circ B$ as closed subsets. Thus $A \circ B - A$ and $A \circ B - B$ form an open covering of $A \circ B$. It is not difficult to see that if $A \circ B$ is locally compact then both A and B must be *compact*. Special cases of the join are cones and suspensions. Properties of the joins of spaces have been studied by numerous authors for various purposes. The present investigation deals with the following question. Which manifolds X can be written as the join $A \circ B$ of two topological spaces? We obtain a rather complete answer to this question in the following.

THEOREM 2. *Let $X = A \circ B$, where A and B are nondegenerate spaces. Then*

1. *X is a combinatorial n-manifold (without boundary) if and only if $X = S^n$, and*

2. *X is a combinatorial n-manifold with nonempty boundary if and only if $X = I^n$.*

This characterizes both S^n and I^n in terms of joins among the combinatorial manifolds. If we assume, as a special case, that both A and B are simplicial and that the join structure of X is that *induced* by the simplicial structures of A and B, then Theorem 1 reduces to a theorem of Alexander [Ann. of Math. **31** (1930), 308]. The point is that if one takes a highest dimensional simplex s of A, then (essentially by the definition of combinatorial manifold) the link of s is combinatorially equivalent to a sphere (or cell). But, the link of s is precisely B by the assumption at hand. We emphasize that we make no such assumptions on A and B and on how the combinatorial structure of X is induced, if at all, from A and B. In fact, neither A nor B need be simplicial or even locally euclidean at any point (see §5).

To prove Theorem 2 one must carefully examine the implications of the locally euclidean structure of X. One discovers that homologically both A and B are

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(2) If $B = \emptyset$, we separately define $A \circ B = \emptyset$. This definition is useful in unifying different cases. See, for instance, Propositions 3.2 and 4.1.

locally like a manifold. This fact together with local homological properties of X imply that globally A and B are sphere- or cell-like.

Now in higher dimensions one uses the truth of the Poincaré conjecture; in lower dimensions one uses the equivalence of homology manifolds with locally euclidean manifolds, and in the intermediary range one gives separate arguments to show that X is actually a cell or a sphere. The homological part of the argument is summarized in

THEOREM 1. *Let* $X = A \circ B$. *Then*

1. *X is a generalized manifold if and only if A and B are sphere-like generalized manifolds* (*and therefore X itself must be a sphere-like generalized manifold*), *and*

2. *X is a generalized manifold with nonempty boundary if and only if either both A and B are generalized cells or one is a generalized cell and the other is a sphere-like generalized manifold* (*and therefore X must be a generalized cell*).

In view of Theorem 1, it is clear that in order to get a generalized manifold by join constructions, one must start with the correct global and *local* homology. (Although the join construction kills or alters global homology the wrong global homology of A and B is retained in the form of the wrong local homology of $A \circ B$.)

Perhaps Theorem 2 could be proved without reference to generalized manifolds with boundary but the steps necessary to avoid the algebraic facts, particularly the handling of the boundary, would be unduly complicated. In §5, where we construct the previously mentioned pathological join factors of S^n and I^n, the use of generalized manifolds with boundary seems unavoidable. For the convenience of the reader, a brief but useful account of the pertinent facts concerning generalized manifolds with boundary is given in §2. §§3 and 4 are devoted to proving Theorems 1 and 2.

2. **Generalized manifolds.** We recall the definition of (*locally orientable*) generalized manifolds or cohomology manifolds in the sense of Wilder [14]. A reference for the definitions and facts stated below as we shall use them is [11]. Let X be a locally compact Hausdorff space. We shall let $H_c^p(X)$, $H^p(X)$, $H_a^p(X)$, denote the Cech cohomology (or equivalently Alexander-Spanier cohomology) with *compact* supports, with *closed* supports and *augmented* cohomology with *closed* supports respectively. The coefficients will always be taken in a principal ideal domain L. The pth *local co-Betti number of X at x is said to be* 0, *if for each open set U containing x there exists an open set V such that $x \in V \subset U$ and the inclusion $i: V \subset U$ induces the trivial homomorphism $i^*: H_c^p(V) \to H_c^p(U)$.*

A locally compact Hausdorff space X is said to be an *orientable generalized n-manifold* over L if

(i) the pth local co-Betti number of X at x is 0, for all $p \neq n$, and all $x \in X$,

(ii) U is any nonempty open connected set of X and C is the component of X containing U, then $i^* : H_c^n(U) \to H_c^n(C)$ is a bijection, and $H_c^n(C) \simeq L$,

(iii) the cohomology dimensions of X with respect to L is finite.

A locally compact Hausdorff space is called a *generalized n-manifold* (n-gm) *over L* if there exists an open covering of X by orientable generalized n-manifolds over L. (This is equivalent to what Wilder calls a locally orientable generalized manifold if L is a field.)

It is easy to see that an n-gm X is orientable if and only if $H_c^n(C) \simeq L$ for each component C of X.

A locally compact Hausdorff space X is called a *generalized n-manifold with boundary B over L* (n-gm with boundary B) if

(i) B is a closed subset of X,

(ii) B is empty or an $(n-1)$-gm over L,

(iii) $X - B$ is an n-gm over L,

(iv) the pth local co-Betti number of X at $x \in B \subset X$ is 0 for all p and all $x \in B \subset X$.

We shall usually supress L, and for an n-gm X with boundary B we usually denote $X - B$ by $\operatorname{Int} X$ and B by $\operatorname{Bd} X$. It is a consequence of [10, 3.32, 3.33, 3.36] and [15, Corollary 5.3] that

2.1. *If A is a locally compact subset of an n-gm X then the interior points of A coincide with the points of A at which the nth local co-Betti number does not vanish. Hence if A is an n-gm with boundary imbedded (as a closed subset) in X then $\operatorname{Int} A$ and $\operatorname{Bd} A$ coincide with the point set interior of A and point set frontier of A. In particular invariance of domain is valid in n-gms.*

A compact n-gm is *sphere-like* if $H^p(X) \simeq H^p(S^n)$, for all p, where S^n denotes the n-sphere. A connected n-gm X is called *euclidean-like* if $H_c^p(X) \simeq H_c^p(E^n)$, for all p, where E^n denotes *n-space*. A compact orientable n-gm X with nonempty boundary is called a *generalized n-cell* if $H_a^p(X) = 0$. It follows, in this last definition, that $\operatorname{Int} X$ is a euclidean-like n-gm and $\operatorname{Bd} X$ is a sphere-like $(n-1)$-gm [8, 1.1].

For the remaining part of this paper we shall restrict ourselves to principal ideal domains which are *fields or the integers.* We recall that in [11, Theorem 6] the following factorization theorem is proved.

2.2. Proposition. *Let $X = A \times B$. Then X is a n-gm with boundary if and only if A is a p-gm with boundary and B is an $(n-p)$-gm with boundary. Furthermore,* $\operatorname{Int} X = \operatorname{Int} A \times \operatorname{Int} B$, $\operatorname{Bd} X = (A \times \operatorname{Bd} B) \cup (\operatorname{Bd} A \times B)$ *where* $(A \times \operatorname{Bd} B) \cap (\operatorname{Bd} A \times B) = \operatorname{Bd} A \times \operatorname{Bd} B$. (*Note that boundaries may be empty.*)

Thus, generalized manifolds form a class of spaces which are closed under factorization, a property unfortunately not enjoyed by classical manifolds. Of course every locally euclidean n-manifold is an n-gm and every separable metric n-gm *is locally euclidean if* $n \leqq 2$. We remark that one equivalent definition of generalized manifolds is that they are the class of spaces for which Poincaré duality holds both locally and globally (possibly with twisted coefficients).

3. **Proof of Theorem 1.** We defined the *join* $A \circ B$ of two spaces A and B in the Introduction. In case B is a single point p, $A \circ p$ is called the *cone* $\mathrm{C}(A)$ over A and p the *vertex* of the cone. $\mathrm{C}(A) - A$ is called the *open cone* over A and denoted by $\mathrm{OC}(A)$. The *suspension* $\mathrm{S}(A)$ of A *is* $A \circ S^0$. We observe that

$$A \circ B = (A \circ B - A) \cup (A \circ B - B)$$

and

$$A \circ B - A = \mathrm{OC}(A) \times B, \qquad A \circ B - B = A \times \mathrm{OC}(B)$$

with

$$((A \circ B) - A) \cap ((A \circ B) - B) = A \times B \times (0,1).$$

Throughout the paper equality means topological equivalence.

The statement and proof of a special case of the following proposition are implicity contained in an earlier paper of the authors [7]. We give the proof here as the present setting is more general and the former setting was not explicit.

3.1. Proposition. *Let* $X = \mathrm{OC}(A)$, $Y = \mathrm{C}(B)$; *then*

1. X *is an* n-gm *if and only if* A *is a sphere-like* $(n-1)$-gm.

2. X *is an* n-gm *with nonempty boundary if and only if* A *is a generalized* $(n-1)$-*cell. Hence* $\mathrm{Bd}\, X = \mathrm{OC}(\mathrm{Bd} A)$, $\mathrm{Int}\, X = A \times E^1$.

3. Y *is an* n-gm *with boundary if and only if* B *is a sphere-like* $(n-1)$-gm *or a generalized* $(n-1)$-*cell. In the former case*, $\mathrm{Bd}\, Y = B$, $\mathrm{Int}\, Y = \mathrm{OC}(B)$, *and in the latter case* $\mathrm{Bd}\, Y = B \cup \mathrm{C}(\mathrm{Bd}\, B)$, $\mathrm{Int}\ Y = \mathrm{Int}\ B \times E^1$.

Proof. Let v denote the vertex of X or Y.

1. First $X - v = A \times E^1$. Thus if X is an n-gm, then A must be an $(n-1)$-gm. Take a conical neighborhood U of v. Then,

$$H_c^p(U) = H^p(X, X - U) = H_a^{p-1}(X - U) = H_a^{p-1}(A) = H_a^{p-1}(S^{n-1}).$$

Conversely, if A is a sphere-like $(n-1)$-gm, then $X - v = A \times E^1$ is an orientable n-gm.

For each conical neighborhood U of v we have $H_c^p(U) = H_a^{p-1}(S^{n-1})$. Thus the pth local co-Betti number of X at v is 0, for all $p \neq n$. Let C be any open connected set containing v. Let U be a conical neighborhood of v contained in C. As long as $n > 1$,

$$i^*: H_c^n(U - x) \simeq H_c^n(U) (\simeq L),$$

which implies that $U - x$ is connected. (In any orientable n-gm the rank of $H_c^n(W)$, where W is an open subset, equals the number of components of W.

This follows directly from Poincaré duality or a little more indirectly from the definitions.) Therefore $C-x$ is connected which implies $i^*:H_c^n(C-x) \simeq H_c^n(X-x)$. Hence, $i^*:H_c^n(C) \simeq H_c^n(X)(\simeq L)$. This shows orientability of X.

2. If v were an interior point then we would be in case 1 and hence X could not have nonempty boundary. Thus v is a boundary point and

$$\mathrm{Bd}\, X = \mathrm{Bd}(X-v) \cup v = \mathrm{Bd}\, A \times E^1 \cup v.$$

Since $\mathrm{OC}(\mathrm{Bd}\, A) = \mathrm{Bd}\, A \times E^1 \cup v$, 1 implies that $\mathrm{Bd}\, A$ is a sphere-like $(n-2)$-gm. All the local co-Betti numbers of X at v vanish and therefore A is an acyclic, connected compact $(n-1)$-gm with sphere-like boundary. Hence A is a generalized $(n-1)$-cell. Conversely let A be a generalized $(n-1)$-cell. Then $\mathrm{OC}(A)-v = A \times E^1$ which is an orientable n-gm with boundary $\mathrm{Bd}\, A \times E^1$. By case 1, $\mathrm{OC}(\mathrm{Bd}\, A)$ is a euclidean-like $(n-1)$-gm. The set $\mathrm{OC}(A) - \mathrm{OC}(\mathrm{Bd}(A)) = \mathrm{Int}\, A \times E^1$ is a euclidean-like n-gm. Since $H_c^p(U) \simeq H_a^{p-1}(A)$, for every conical neighborhood U of v it follows that condition (iv) of the definition of an n-gm with boundary is satisfied.

3. Write $Y = \mathrm{C}(B)$ as the union of two *open* sets $Y - B = \mathrm{OC}(B)$ and $B \times [0,1)$. Thus Y is an n-gm with boundary if and only if $\mathrm{OC}(B)$ and $B \times [0,1)$ are n-gms with boundary.

($\mathrm{Bd}\, Y = \mathrm{Bd}(\mathrm{OC}(B)) \cup \mathrm{Bd}(B \times [0,1))$.) We can apply cases 1 and 2 to $\mathrm{OC}(B)$ and see that B must be either a sphere-like or cell-like $(n-1)$-gm. In case B is a sphere-like $(n-1)$-gm, B is the boundary of Y since it is the boundary of $[0,1) \times B$, and therefore Y is a generalized n-cell with B as boundary. In case B is a generalized $(n-1)$-cell, $Y - v = B \times [0,1)$ has boundary $(\mathrm{Bd}\, B \times [0,1)) \cup (B \times 0)$. Thus $\mathrm{Bd}\, Y = B \cup \mathrm{C}(\mathrm{Bd}\, B)$ is a sphere-like $(n-1)$-gm since $\mathrm{C}(B)$ is acyclic.

Proof of Theorem 1. Let $X = A \circ B$. Then X is the union of the two subsets $\mathrm{OC}(A) \times B$, $A \times \mathrm{OC}(B)$. Propositions 2.2 and 3.1 imply (a) X is an n-gm if and only if A and B are sphere-like p and $(n-p-1)$-gms; (b) X is an n-gm with boundary if and only if A and B are generalized cells or one is a generalized cell and the other a sphere-like generalized manifold of the correct dimensions. It just remains to show that in (a) $H^p(X) = H^p(S^n)$ and in (b) $H_a^p(X) = 0$. *For any join $X = A \circ B$, the sequence of reduced cohomology groups*

$$0 \to H_a^p(\mathrm{C}(A) \times B) \oplus H_a^p(A \times \mathrm{C}(B)) \to H_a^p(A \times B \times \tfrac{1}{2}) \xrightarrow{d^*} H_a^{p+1}(A \circ B) \to 0$$

is exact. For since $X = (\mathrm{C}(A) \times B) \cup (A \times \mathrm{C}(B))$ such that

$$(\mathrm{C}(A) \times B) \cap (A \times \mathrm{C}(B)) = (A \times B \times \tfrac{1}{2})$$

one may consider the associated Mayer-Vietoris exact sequence of reduced cohomology groups. Observe that $\mathrm{C}(A) \times B$ and $A \times \mathrm{C}(B)$ are contractible in $A \circ B$, thus $H^p(A \circ B) \to H^p(\mathrm{C}(A) \times B) \oplus H^p(A \times \mathrm{C}(B))$ is always trivial which yields the exact sequence above. In case X is compact we may apply the Künneth theorem to $H^p[A \times B \times \frac{1}{2}]$. If $H^*(A)$ and $H^*(B)$ are torsion free the exact sequence above becomes the exact sequence

$$0 \to \sum_{r+s=p} H_a^r(A) \otimes H_a^s(B) \xrightarrow{d^*} H_a^{p+1}(A \circ B) \to 0$$

of *reduced cohomology groups* (cf. [9]). Thus, if either A or B is acyclic then so is $A \circ B$; if both A and B are sphere-like then so is $A \circ B$. This completes the proof of Theorem 1.

As a corollary to Theorem 1 we obtain

3.2. PROPOSITION. *Let $X = A \circ B$ be an n-*gm *with boundary (possibly empty). Then*

$$\mathrm{Bd}\, X = \mathrm{Bd}\, A \circ B \cup A \circ \mathrm{Bd}\, B,$$

where

$$\mathrm{Bd}\, A \circ B \cap A \circ \mathrm{Bd}\, B = \mathrm{Bd}\, A \circ \mathrm{Bd}\, B.$$

Proof. As before we can write X as the union of the open n-gms with boundary $\mathrm{OC}(A) \times B, A \times \mathrm{OC}(B)$. Thus

$$\mathrm{Int}\, X = \mathrm{Int}(\mathrm{OC}(A) \times B) \cup \mathrm{Int}(A \times \mathrm{OC}(B)),$$

and

$$\mathrm{Bd}\, X = \mathrm{Bd}(\mathrm{OC}(A) \times B) \cup \mathrm{Bd}(A \times \mathrm{OC}(B)).$$

Since $\mathrm{Bd}(\mathrm{OC}(A) \times B)$ and $\mathrm{Bd}(A \times \mathrm{OC}(B))$ are open subsets of $\mathrm{Bd}\, X$ it follows that $\mathrm{Bd}(\mathrm{OC}(A) \times B) \cap \mathrm{Bd}(A \times \mathrm{OC}(B))$ is an $(n-2)$-gm with boundary. If $\mathrm{Bd}\, X$ is empty then both A and B must be sphere-like and the proposition is clear in this case. If $\mathrm{Bd}\, X$ is not empty then at least A or B is cell-like. Let us examine the case where A is cell-like and B is sphere-like.

$$\mathrm{Int}(\mathrm{OC}(A) \times B) = \mathrm{Int}(\mathrm{OC}(A)) \times \mathrm{Int}\, B = \mathrm{Int} A \times E^1 \times B,$$

$$\mathrm{Int}(A \times \mathrm{OC}(B)) = \mathrm{Int}\, A \times \mathrm{Int}(\mathrm{OC}(B)).$$

$$\mathrm{Bd}(\mathrm{OC}(A) \times B) = \mathrm{Bd}(\mathrm{OC}(A)) \times B \cup \mathrm{OC}(A) \times \mathrm{Bd}\, B = \mathrm{OC}(\mathrm{Bd}\, A) \times B,$$

$$\mathrm{Bd}(A \times \mathrm{OC}(B)) = \mathrm{Bd}\, A \times \mathrm{OC}(B) \cup A \times \mathrm{Bd}(\mathrm{OC}(B)) = \mathrm{Bd}\, A \times \mathrm{OC}(B),$$

hence $\mathrm{Bd}(A \circ B) = \mathrm{Bd}\, A \circ B$. Now supposing both A and B are cell-like we have that $\mathrm{Bd}(A \circ B)$ is the union of the open (in $\mathrm{Bd}(A \circ B)$) subsets

$$\mathrm{Bd}(\mathrm{OC}(A) \times B) = \mathrm{Bd}\,\mathrm{OC}(A) \times B \cup \mathrm{OC}(A) \times \mathrm{Bd}\, B = \mathrm{OC}(\mathrm{Bd} A) \times B \cup \mathrm{OC}(A) \times \mathrm{Bd} B,$$

$$\mathrm{Bd}(A \times \mathrm{OC}(B)) = \mathrm{Bd} A \times \mathrm{OC}(B) \cup A \times \mathrm{Bd}(\mathrm{OC}(B)) = \mathrm{Bd} A \times \mathrm{OC}(B) \cup A \times \mathrm{OC}(\mathrm{Bd}(B)).$$

Putting these two sets together we get $\mathrm{Bd}(A \circ B) = \mathrm{Bd}\, A \circ B \cup A \circ \mathrm{Bd}\, B$. Furthermore $(\mathrm{Bd}\, A \circ B) \cap (A \circ \mathrm{Bd}\, B) = \mathrm{Bd} A \circ \mathrm{Bd}\, B$ which is the part common to both parts.

4. Proof of Theorem 2.

4.1. PROPOSITION. *$A \circ B$ is locally contractible if and only if A and B are locally* *contractible. If $A \circ B$ is an n-*gm *with locally contractible boundary then* BdA *and* BdB *are also locally contractible.*

Proof. $A \circ B$ is the union of the open subsets $A \times \mathrm{OC}(B)$ and $\mathrm{OC}(A) \times B$. Hence $A \circ B$ is locally contractible if and only if A and B are. Now suppose $A \circ B$ is an n-gm with locally contractible boundary. Since

$$\mathrm{Bd}(A \circ B - A) = \mathrm{Bd}(\mathrm{OC}(A) \times B) = \mathrm{BdOC}(A) \times B \cup (\mathrm{OC}(A) \times \mathrm{Bd}B)$$

is an open subset of $\mathrm{Bd}(A \circ B)$, $\mathrm{Int}(\mathrm{OC}(A)) \times \mathrm{Int}(\mathrm{Bd}\,B)$ is an open subset of $\mathrm{Bd}(A \circ B)$ and hence locally contractible. Therefore, $\mathrm{Int}(\mathrm{Bd}\,B) = \mathrm{Bd}\,B$ is locally contractible. Similarly, $\mathrm{Bd}A$ is locally contractible.

4.2. Proposition. *Let* $\mathrm{S}(X)$ *be an n-manifold with nonempty boundary. Then* $\mathrm{S}(X) = I^n$.

Proof. X is a generalized $(n-1)$-cell by Theorem 1. By Proposition 3.2, $\mathrm{Bd}(\mathrm{S}(X)) = \mathrm{S}(\mathrm{Bd}X)$. On the other hand, any manifold that is the suspension of a space is easily seen to be covered by two open cells. Hence the generalized Schoenflies theorem [4] can be used to show that it is a sphere. So, $\mathrm{Bd}(\mathrm{S}(X)) = S^{n-1}$. However, either [6] or the proposition that follows implies that $\mathrm{Int}(\mathrm{S}(X)) = E^n$. Since $\mathrm{Bd}(\mathrm{S}(X))$ is collared in $\mathrm{S}(X)$ [5], $\mathrm{S}(X)$ is the union of an n-cell and an n-annulus around it. Thus $\mathrm{S}(X) = I^n$.

4.3. Proposition. *If* $\mathrm{OC}(X)$ *is an n-manifold with nonempty boundary, then* $\mathrm{OC}(X) = E^{n-1} \times [0, \infty)$.

Proof. We will consider $\mathrm{OC}(X)$ as $X \times (0,1]$ with $X \times 1$ identified to the vertex $\hat{p}$. For each $x \in X$, the set

$$r_x = \{(x,t) \mid 0 < t \leqq 1\}$$

is called the *ray toward* x and t in (x,t) is called the *ray* coordinate. For each e, $0 < e < 1$, L_e will denote the subset of $\mathrm{OC}(X)$ consisting of the points with ray coordinates at least e. For convenience we define

$$H_e = \{x \mid x \in X,\ \rho(x, \mathrm{Bd}X) \geqq e\}$$

and

$$K_e = \bigcup_{x \in H_e} r_x .$$

Since $\mathrm{Bd}(\mathrm{OC}(X)) = \mathrm{OC}(\mathrm{Bd}\,X)$ is homeomorphic to $\mathrm{S}(\mathrm{Bd}\,X)$ minus a point, it follows that $\mathrm{Bd}(\mathrm{OC}(X))$ is homeomorphic to E^{n-1}. Hence there exists a homeomorphism h of E^{n-1} onto $\mathrm{Bd}(\mathrm{OC}(X))$.

Let $C_1, C_2, \cdots$ be the subsets of E^{n-1} defined by $C_i = \{x \in E^{n-1} \mid \|x\| \leqq i\}$.

Since $\mathrm{Bd}(\mathrm{OC}(X))$ is collared in $\mathrm{OC}(X)$, there exists a homeomorphism

$$k : \mathrm{Bd}(\mathrm{OC}(X)) \times [0,2] \to \mathrm{OC}(X)$$

such that $k(x,0) = x$ for each $x \in \mathrm{Bd}(\mathrm{OC}(X))$.

We are now ready to construct certain n-cells $E_1, E_2, \cdots$ such that $\bigcup E_i = \mathrm{OC}(X)$.

1. *Construction of* E_1. Let i_1 be an integer such that $h(C_{i_1})$ contains $\mathrm{Bd}(\mathrm{OC}(X)) \cap L_{1/2}$ in its interior. There exists a sufficiently small positive number e such that

$$L_{1/2} - K_e \subset k(h(C_{i_1}) \times [0,1]).$$

Let $d > \frac{1}{2}$ be a number smaller than 1 such that

$$k(h(C_{i_1}) \times [0,1]) \supset L_d.$$

Since X is normal, there exists a continuous map

$$\phi : X \to [0,1]$$

such that $\phi(\mathrm{Bd}\, X) = 0$ and $\phi(H_e) = 1$.

We now define a homeomorphism f_x of r_x onto itself which, in terms of ray coordinates, takes

$$(0,d] \text{ proportionally onto } (0, d - \phi(x)(d - \tfrac{1}{2})]$$

and

$$[d,1] \text{ proportionally onto } [d - \phi(x)(d - \tfrac{1}{2}), 1].$$

The totality of f_x generates a homeomorphism f_1 of $\mathrm{OC}(X)$ onto itself which leaves invariant each point of $\mathrm{Bd}(\mathrm{OC}(X))$ and stretches the n-cell $k(h(C_{i_1}) \times [0,1])$ so that

$$E_1 = f_1(k(h(C_{i_1}) \times [0,1])) \supset L_{1/2}.$$

2. *Construction of* E_2. There exists an integer j_2 such that $L_{1/(j_2 - 1)} \supset E_1$. Let i_2 be an integer such that $h(C_{i_2})$ contains $\mathrm{Bd}(\mathrm{OC}(X)) \cap L_{1/j_2}$ in its interior. Consider the n-cell $k(h(C_{i_2}) \times [0,1])$, as in 1. There exists a homeomorphism f_2 of $\mathrm{OC}(X)$ onto itself which leaves invariant each point of $\mathrm{Bd}(\mathrm{OC}(X))$ and such that

$$E_2 = f_2(k(h(C_{i_2}) \times [0,1])) \supset L_{1/j_2}.$$

We similarly construct $E_3, E_4, \cdots$.

Then $\bigcup E_i = \mathrm{OC}(X)$. Now observe that $E_1' = \mathrm{Cl}(E_2 - E_1)$ is an n-cell and $E_1 \cap E_1'$ is an $(n-1)$-cell F_1 such that $\mathrm{Bd}F_1$ is nicely imbedded in each of $\mathrm{Bd}E_1$ and $\mathrm{Bd}E_1'$ (see [5]). Hence there exists a homeomorphism of E_2 onto the unit ball in E^n which takes E_1 onto the upper half of the unit ball. We note that each E_i is situated in E_{i+1} in the same way as E_1 is situated in E_2.

Now there exists a homeomorphism h_1 of $C_{i_1} \times [0,1]$ onto E_1 such that $h_1 | C_{i_1} = h | C_{i_1}$. Here we suppose E^{n-1} is naturally imbedded in E^n. There exists a homeomorphism h_2 of $C_{i_2} \times [0,2]$ onto E_2 such that $h_2 | C_{i_2} = h | C_{i_2}$ and $h_2 | C_{i_1} \times [0,1] = h_1$. We similarly construct $h_3, h_4, \cdots$ which together generate a homeomorphism of $E^{n-1} \times [0, \infty)$ onto $\mathrm{OC}(X)$.

4.4. Corollary. *Let M be a triangulated n-manifold with boundary and v a vertex on* BdM. *Then the open star of v is homeomorphic to the n-cell minus a boundary point.*

This fact is well known for combinatorial triangulations. B. Mazur and R. Rosen have proved an analogous result for triangulated manifolds *without* boundary.

Remark. Proposition 4.3 and Corollary 4.4 are interesting since it is not true in general (see 5.6) that a manifold with boundary is $E^{n-1} \times [0, \infty)$ if the interior is E^n and the boundary is E^{n-1}.

We are now ready for the

Proof of Theorem 2. Let $X = A \circ B$. The "if" part is clear. We prove the "only if" part.

Case 1. $\mathrm{Bd}(A \circ B) = \varnothing$. If $n \leqq 6$, at least one of A and B is an actual sphere. Hence by the associativity of the join operation, $X = \mathrm{S}(Y)$ for some space Y. Hence $X = S^n$. If $n \geqq 7$, X is by Theorem 1, an integral cohomology sphere. Hence it is an integral homology sphere in the sense of singular (or simplicial) homology. On the other hand $\pi_1(X) = 1$. This fact follows from Lemma 2.2 of [8] or the van Kampen theorem applied to $A \circ B = (A \circ B - A) \cup (A \circ B - B)$. Hence by the generalized Poincaré conjecture [12; 13; 16], $X = S^n$.

Case 2. $\mathrm{Bd}(A \circ B) \neq \varnothing$. If $n \leqq 6$, at least one of A and B is an actual cell or sphere of dimension $r \leqq 2$ and if $r = 0$, it is S^0 by the nondegeneracy requirement of the theorem. Hence again $X = \mathrm{S}(Y)$ for some space Y and $X = I^n$ by Proposition 4.2. If $n \geqq 7$, we prove that X is contractible and BdX is S^{n-1}. Then the generalized Poincaré conjecture and the generalized Schoenflies theorem imply that $X = I^n$.

(i) X is contractible, $\pi_1(X) = 1$ by Lemma 2.2 of [8]. By Theorem 1, X is homologically trivial.

(ii) $\mathrm{Bd}X = S^{n-1}$. Since X is a generalized n-cell by Theorem 2, BdX is a combinatorial $(n-1)$-manifold with sphere-like cohomology and hence sphere-like homology. By the generalized Poincaré conjecture, it suffices to show $\pi_1(\mathrm{Bd}X) = 1$. But

$$\mathrm{Bd}X = \mathrm{Bd}A \circ B \cup A \circ \mathrm{Bd}B$$

with

$$\mathrm{Bd}A \circ B \cap A \circ \mathrm{Bd}B = \mathrm{Bd}A \circ \mathrm{Bd}B.$$

All three joins considered are simply connected and locally contractible and therefore by van Kampen's theorem, $\pi_1(\mathrm{Bd}X) = 1$.

Remark. From the proof, it is obvious that the nondegeneracy requirement is *not* necessary except perhaps for the case where $\mathrm{Bd}X \neq \varnothing$ and $n = 4$ or 5.

5. **Manifolds modulo arcs.** We have shown that among the combinatorial manifolds, essentially only S^n and I^n can be expressed as joins. However, one should not expect nice join factors for S^n and I^n. Proposition 5.1 gives, in addition to its positive nature, the method to construct pathological join factors.

Let X^p and Y^p be obtained from I^p and S^p, respectively, by shrinking each of a finite number of arcs in its interior to a point. By [1],

$$X^p \times E^1 = I^p \times E^1$$

and

$$Y^p \times E^1 = S^p \times E^1.$$

By taking two-point compactifications,

$$\mathrm{S}(X^p) = I^{p+1} \text{ and } \mathrm{S}(Y^p) = S^{p+1}.$$

Hence by Proposition 4.3 and the proof of Proposition 4.2,

$$\mathrm{OC}(X^p) = E^p \times [0, \infty) \text{ and } \mathrm{OC}(Y^p) = E^{p+1}.$$

On the other hand, it is known [8] that

$$X^p \times I^2 = I^p \times I^2$$

and therefore, $Y^p \times I^2$ is a manifold with boundary. Hence if M is a manifold with boundary of dimension $\geqq 2$, then $X^p \times M$ and $Y^p \times M$ are all manifolds with boundary. Let Z^p denote either one of X^p and Y^p. Then

$$Z^p \circ Z^q = Z^p \times \mathrm{OC}(Z^q) \cup \mathrm{OC}(Z^p) \times Z^q$$

and the two sets on the right are open in $Z^p \circ Z^q$. Hence

If $p, q \geqq 1$, $Z^p \circ Z^q$ is a manifold with boundary.

5.1. Proposition. *Let $p, q \geqq 1$ and X^p, etc., as above. Then*

1. $X^p \circ X^q = I^{p+q+1}$,
2. $X^p \circ Y^q = I^{p+q+1}$,
3. $Y^p \circ Y^q = S^{p+q+1}$.

Proof. We first note all the joins here are manifolds with boundary. We also recall that I^{p+q+1} is characterized by the fact that it is a manifold with boundary whose interior and boundary are respectively E^{p+q+1} and S^{p+q}.

We apply induction on p. If $p \leqq 2$, then the proposition is true. Suppose it has been proved for all $p < r$. It suffices to prove the case $p = r$, which we will prove by induction on q. If $q \leqq 2$, the proposition is true. Suppose it has been proved for all $q < s$. The proof will be completed if we prove for $p = r$, $q = s$ assuming that it is true for all p, q such that $p < r$ or $p = r$ and $q < s$.

1. $\mathrm{Bd}(X^r \circ X^s) = \mathrm{Bd}X^r \circ X^s \cup X^r \circ \mathrm{Bd}X^s = S^{r-1} \circ X^s \cup X^r \circ S^{s-1}$ where

$$S^{r-1} \circ X^s \cap X^r \circ S^{s-1} = S^{r-1} \circ S^{s-1} = S^{r+s-1} = \mathrm{Bd}(S^{r-1} \circ X^s) = \mathrm{Bd}(X^r \circ S^{r-1}).$$

Hence $\mathrm{Bd}(X^r \circ X^s) = S^{r+s}$.

On the other hand,

$$\mathrm{Int}(X^r \circ X^s) = \mathrm{Int}\, X^r \times \mathrm{Int}\, X^s \times E^1 = \mathrm{Int}\, X^r \times E^{s+1} = E^{r+s+1}.$$

2. $\mathrm{Bd}(X^r \circ Y^s) = \mathrm{Bd} X^r \circ Y^s = S^{r-1} \circ Y^s = S^{r+s}$. Therefore, $Y^s \subset \mathrm{Bd}(X^r \circ Y^s)$ and

$$\mathrm{Int}(X^r \circ Y^s) \subset \mathrm{Int}(X^r \circ Y^s - Y^s) = \mathrm{Int}(X^r \times \mathrm{OC}(Y)).$$

Since $\mathrm{Int}(X^r \circ Y^s) \supset \mathrm{Int}(X^r \times \mathrm{OC}(Y^s))$,

$$\mathrm{Int}(X^r \circ Y^s) = \mathrm{Int}(X^r \times \mathrm{OC}(Y^s)) = \mathrm{Int}\, X^r \times E^{s+1} = E^{r+s+1}.$$

3. Let $Y^r = X_1^r \cup X_2^r$, where X_1^r, X_2^r are some X^r and

$$\mathrm{Bd} X_1^r \cap \mathrm{Bd} X_2^r = X_1^r \cap X_2^r = \mathrm{Bd} X_1^r = \mathrm{Bd} X_2^r = S^{r-1}.$$

Then $Y^r \circ Y^s = (X_1^r \circ Y^s) \cup (X_2^r \circ Y_s^s)$ with

$$(X_1^r \circ Y^s) \cap (X_2^r \circ Y^s) = \mathrm{Bd}(X_1^r \circ Y^s) = \mathrm{Bd}(X_2^r \circ Y^s) = S^{r-1} \circ Y^s = S^{r+s}.$$

Since $X_1^r \circ Y^s = I^{r+s+1}$ by 2 above,

$$Y^s \circ Y^s = S^{s+s+1}.$$

This completes the proof.

We remark that Proposition 5.1 can be used to show that for $n \geqq 7$, I^n and S^n can be expressed as the join of two spaces A and B where neither is a manifold with boundary. In fact, A and B can be constructed so that neither one is locally euclidean at any point.

We proved Proposition 5.1 for $p, q \geqq 1$. However, it has been already pointed out that

5.2. PROPOSITION.

$$\mathrm{S}(X^p) = I^{p+1}, \ \mathrm{S}(Y^p) = S^{p+1},$$

$$\mathrm{OC}(X^p) = E^p \times [0, \infty), \ \mathrm{OC}(Y^p) = E^{p+1}.$$

The similar results for the cone are not true. In fact, for $p=3$, no modifications of X^3 makes an analogous statement valid as the following proposition shows.

5.3. PROPOSITION. $\mathrm{C}(X) = I^4$ *implies* $X = I^3$.

Proof. X is a generalized 3-cell by Theorem 1 and hence $\mathrm{Bd}(X) = S^2 \subset \mathrm{Bd} I^4 = S^3$. That $\mathrm{Int}\, X$ is 1-ULC follows exactly the same way as in [2]. Hence by [3], $X = I^3$.

However if we let $M^q = X^{q-1} \times I$, then though M^q is not a manifold with boundary in general, we have

5.4. PROPOSITION. $\mathrm{S}(M^q) = \mathrm{C}(M^q) = I^{q+1}$.

Proof. $\mathrm{S}(M^q)$ is a two-point compactification of $M^q \times E^1$. Observe that

$M^q \times E^1$ has exactly two ends and therefore the two-point compactification is unique. On the other hand,

$$M^q \times E^1 = X^{q-1} \times I \times E^1 = X^{q-1} \times E^1 \times I = \iota^{q-1} \times E^1 \times I = I^q \times E^1.$$

Hence $\mathrm{S}(M^q) = \mathrm{S}(I^q) = I^{q+1}$.

$\mathrm{S}(M^q)$ is obtained from $I^{q+1} = M^q \times I$ by shrinking each of $M^q \times 0$ and $M^q \times 1$ to a point. This shows that $M^q \times 0$ and $M^q \times 1$ are cellular (in the sense of Brown [4]) on $\mathrm{Bd}I^{q+1}$. Then it follows that $\mathrm{Bd}(\mathrm{C}(M^q)) = \mathrm{S}^q$. On the other hand, $\mathrm{Int}(\mathrm{C}(M^q)) = \mathrm{Int}\, M^q \times E^1 = E^{q+1}$. Finally $\mathrm{C}(M^q)$ is locally like either $\mathrm{S}(M^q)$ or $M^q \times I$. Hence it is a manifold with boundary. These together show that $\mathrm{C}(M^q) = I^{q+1}$.

In connection with cones and suspensions, we observe the following.

5.5. Proposition. *Let $X \times I = I^{q+1}$, where X^q is a manifold with boundary. Then $\mathscr{D}(\mathrm{C}(X^q)) = S^{q+1} = \mathscr{D}(\mathrm{S}(X^q))$, where $\mathscr{D}$ denotes the double which is the union of two copies of a* gm *along their boundaries.*

Proof. Using the collaredness of $\mathrm{Bd}X$ in X, we find that $\mathscr{D}(X) = \mathrm{Bd}(X \times I) = S^q$. This implies that $X^q \times 1$ and $X^q \times 0$ are cellular in $S^{p+1} = \mathscr{D}(X^q \times I)$. Hence if we shrink one or both of $X^q \times 1$ and $X^q \times 0$. we obtain S^{p+1}. On the other hand, it is $\mathscr{D}(\mathrm{S}(X^q))$ or $\mathscr{D}(\mathrm{C}(X^q))$ as the case may be.

Proposition 5.5 reveals peculiar decompositions of S^{q+1}.

5.6. We append below the example of a 3-manifold X such that $\mathrm{Int}\,X = E^3$, $\mathrm{Bd}X = E^2$ but $X \neq E^2 \times [0,\infty)$ (see 4.3).

Let D be a disk and p the vertex of the cone $\mathrm{C}(D)$. Consider a cone segment $\mathrm{C}(x)$, $x \in \mathrm{Bd}D$. There exists a homeomorphism h of $\mathrm{C}(D)$ into E^3 such that $h(\mathrm{C}(x))$ is the wild arc 1.2 of Artin and Fox [Ann. of Math.(2) **49** (1948), 979–990], $h(p) = q$, $h(\mathrm{Bd}\,\mathrm{C}(D))$ is locally tame except at q and $E^3 - h(\mathrm{C}(D))$ is homeomorphic to the complement of the (wild) arc and therefore is an open 3-cell. Then compactify E^3 at infinity to obtain S^3. Let $X \times \mathrm{Cl}(S^3 - h(\mathrm{C}(D))) - q$. Then

1. $\mathrm{Bd}X = E^2$,
2. $\mathrm{Int}\,X = S^3 - h(\mathrm{C}(D)) = E^3$,
3. X is a manifold with boundary, and
4. $X \neq E^2 \times [0,\infty)$.

1–3 are obvious. If $X = E^2 \times [0,\infty)$, its one-point compactification $\mathrm{Cl}(\mathrm{S}^3 - h(\mathrm{C}(D))) = I^3$ and therefore $\mathrm{Bd}\,\mathrm{C}(D)$ would be a tame 2-sphere and the arc $h(\mathrm{C}(X))$ on the tame sphere would be tame.

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